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On the one-point Friedrichs model

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Abstract. We investigate both analytically and numerically the one-point Friedrichs model characterised by a λV interaction with a non-square integrable coupling function $v(\omega) = 1$. If the continuous spectrum of the Hamiltonian is bounded from below, a form factor is needed to avoid some infinities and the solution of the spontaneous emission problem then depends on an unphysical cut-off parameter α. This solution is denoted by $p(t; \alpha)$ and its α-dependence may be removed by taking the appropriate limit. Indeed, we find that: $\mathrm{Lim}_{\alpha \to \infty} p(t; \alpha) = 1$. If, on the contrary, the continuous spectrum is assumed to be infinite, then no form factor is needed and $p(t) = \exp(-2\pi\lambda^2 t)$.

1. Introduction

The finiteness of Weisskopf–Wigner (WW) type theories describing the interaction of an 'm-level' atom ($m < +\infty$) with a bounded number of photons, has been rigorously demonstrated under some weak conditions (essentially that the kernels of some integral operators have to be square integrable) by Grimm and Ernst (1974). These conditions are fulfilled when the atom is either the Dirac (Grimm and Ernst 1974) or the non-relativistic (Moses 1973, Davidovich and Nussenzveig 1980) hydrogen atom. Unfortunately, once the well known dipole approximation is used, some kernels are no longer square integrable and some infinities appear (Grimm and Ernst 1974, 1975). Within the lowest order WW theory (a two-level atom in interaction with one photon only), Grimm and Ernst (1975) showed in particular why a combination of the dipole approximation with a 'physical' cut-off at distances of the order of the inverse Bohr radius leads to realistic results on the line shifts, etc. Similar conclusions were reached by Davidovich and Nussenzveig (1980). However, from the 'mathematical' point of view one may combine the dipole approximation with *any* form factor which eliminates these infinities. An extensive (numerical) study of the influence of form factors on the dynamics (of spontaneous emission) has been done by Yang *et al* (1974). These authors considered a two-level atom in interaction with a one-dimensional radiation field.

In this paper we consider the lowest order WW theory (Grimm and Ernst 1975). It coincides (Grecos 1978) with the one-point Friedrichs (1948) model. We assume the radiation field to be one dimensional. The Hamiltonian is represented by a 'matrix', one of its 'elements' is an integral operator with the coupling function $v(\omega)$ as kernel. If $v(\omega)$ is not square integrable, a form factor is (in general) needed to avoid some infinities. The solution to the spontaneous emission problem then depends on an unphysical cut-off parameter α. Once this solution (denoted hereafter by $p(t;\alpha)$) has been computed, one may remove its α-dependence by taking the appropriate limit $\alpha\to+\infty$. We shall show that, for the case of a constant coupling function $v(\omega)=1$, no form factor is needed if the continuous part of the spectrum of the Hamiltonian is infinite: $\omega\in(-\infty;+\infty)$. The solution of the spontaneous emission problem then decays exponentially: $p(t)=\exp(-2\pi\lambda^2 t)$. If, on the contrary, the continuous spectrum is bounded from below, a form factor is needed and $\mathrm{Lim}_{\alpha\to\infty}\, p(t;\alpha)=1$.

This paper is divided as follows. In § 2, we recall briefly the one-point Friedrichs model and the solution of the initial value problem. In § 3, we consider an interaction with $v(\omega)=1$. Two cases are considered: the continuous spectrum of the Hamiltonian is (a) infinite, (b) semi-infinite. A numerical investigation of the solution of the spontaneous emission problem (in particular its α-dependence) is presented in § 4. We also analyse its dependence on the lower end-point of the continuous spectrum and on the coupling parameter. Conclusions and some comment for an interaction with $v(\omega)\simeq\omega^{1/2}$ for large values of ω are given in § 5.

2. The one-point Friedrichs model

The one-point Friedrichs model consists of an unperturbed Hamiltonian, H_0, with an absolutely continuous spectrum extending over some interval $\mathscr{I}$ of the real axis and a point eigenvalue ω_0, embedded in it. A λV interaction (λ being a real coupling parameter) couples the point eigenvalue to the continuum.

In the spectral representation of H_0, the total Hamiltonian, $H=H_0+\lambda V$, is given by the 'matrix'

$$H\Leftrightarrow\begin{pmatrix}\omega_0 & \lambda\bar{v}(\omega') \\ \lambda v(\omega) & \omega\delta(\omega-\omega')\end{pmatrix}\tag{2.1}$$

and an element $|f\rangle$ of the Hilbert space $\mathscr{H}$ on which H acts is represented by a column vector

$$|f\rangle\Leftrightarrow\{f_0;f(\omega)\}.\tag{2.2}$$

Here f_0 is a complex number, $f(\omega)$ is a square integrable function, and $\delta(\omega-\omega')$ is the usual Dirac functional. The Hilbert space $\mathscr{H}$ is equipped with the scalar product

$$\langle g|f\rangle=\bar{g}_0f_0+\int_{\mathscr{I}}\mathrm{d}\omega'\ \bar{g}(\omega')f(\omega').\tag{2.3}$$

The element $H|f\rangle$ is represented by

$$H|f\rangle\Leftrightarrow\left\{\omega_0f_0+\lambda\int_{\mathscr{I}}\mathrm{d}\omega'\ \bar{v}(\omega')f(\omega');\ \lambda v(\omega)f_0+\omega f(\omega)\right\}.\tag{2.4}$$

Once an initial condition $|f(t=0)\rangle\equiv|g\rangle$ is given, the time evolution of the state $|f(t)\rangle$ is completely determined by the Schrödinger equation

$$\mathrm{i}\partial_t|f(t)\rangle=H|f(t)\rangle.\tag{2.5}$$

The probability for the initial state $|g\rangle$ to survive is $(t \geqslant 0)$

$$\begin{aligned} p(t) &= |\langle g|f(t)\rangle|^2 = |\langle g|\, e^{-iHt} g\rangle|^2 \\ &= \left| \frac{1}{2\pi i} \int_{\bar{C}} dz\, e^{-izt} \left\langle g \left| \frac{1}{H-z} g \right\rangle \right.\right|^2 \end{aligned} \tag{2.6}$$

where $\bar{C}$ is a straight line parallel to the real axis, from above.

For models of the Friedrichs type, it is easy to compute the resolvant of H (see for example, Marchand 1968, Grecos 1978). We have $(z \notin sp\, H)$

$$\frac{1}{H-z} \Leftrightarrow \frac{1}{\eta(z)} \begin{pmatrix} 1 & \dfrac{-\lambda \bar{v}(\omega')}{\omega' - z} \\ \dfrac{-\lambda v(\omega)}{\omega - z} & \dfrac{\eta(z)\delta(\omega-\omega')}{\omega - z} + \dfrac{\lambda^2 v(\omega)\bar{v}(\omega')}{(\omega-z)(\omega'-z)} \end{pmatrix} \tag{2.7}$$

with

$$\eta(z) = \omega_0 - z - \lambda^2 \sigma(z) \tag{2.8}$$

$$\sigma(z) = \int_{\mathscr{I}} d\omega |v(\omega)|^2/(\omega - z). \tag{2.9}$$

The function $\eta(z)$ is analytic in the complex plane except for a cut $\mathscr{I}$ along the real axis. Its possible zero(s) are real, simple, isolated and lie outside the cut if $v(\omega) \neq 0$ for all $\omega \in \mathscr{I}$ (Friedrichs 1948, Marchand 1968, Grecos 1978). A zero of $\eta(z)$ is a pole of the resolvant of H and thus an eigenvalue of H.

For the systems of physical interest, the continuous spectrum of H is bounded from below. The dipole approximation leads to an interaction with $v(\omega) \simeq \omega^{1/2}$ for large values of ω (Grimm and Ernst 1975). Sometimes one assumes an interaction with $v(\omega) = \omega_0^{1/2}$ (Davidson and Kozak 1970). In both cases the kernel $v(\omega)$ is not square integrable, equation (2.9) diverges (in general) and a form factor is needed.

We therefore introduce a form factor $\chi(\omega; \alpha)$ and we require $\chi(\omega; \alpha)$ to be such that

(a) $\chi(\omega; \alpha) > 0$ for all $\omega \in \mathscr{I}$, except at the (lower) end-point of $\mathscr{I}$ where $\chi(\omega; \alpha)$ may vanish.

(b) $\mathrm{Lim}_{\alpha \to \infty} \chi(\omega; \alpha) = 1$ for all $\omega \in \mathscr{I}$, probably except at the (lower) end-point of $\mathscr{I}$.

(c) $\int_{\omega_*}^{+\infty} d\omega\, \chi(\omega; \alpha)|v(\omega)|^2/\omega < +\infty \qquad (\omega_* > 0).$

Here α represents an unphysical cut-off parameter. Then we replace the 'old' complex function $\eta(z)$ (see (2.8)) by the 'new' one

$$\eta(z; \alpha) = \omega_0 - z - \lambda^2 \sigma(z; \alpha) \tag{2.10}$$

with

$$\sigma(z; \alpha) = \int_{\mathscr{I}} d\omega\, \chi(\omega; \alpha)|v(\omega)|^2/(\omega - z). \tag{2.11}$$

For the spontaneous emission problem, the initial condition is $|g\rangle \Leftrightarrow \{1; 0\}$ and the survival probability becomes ($t \geqslant 0$)

$$p(t;\alpha) = |f_0(t;\alpha)|^2 = \left| \frac{1}{2\pi \mathrm{i}} \int_{\bar{C}} \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}zt} \frac{1}{\eta(z;\alpha)} \right|^2 . \tag{2.12}$$

3. An interaction with $v(\omega) = 1$

We consider the case of a constant coupling function $v(\omega) = 1$. As mentioned before, the continuous spectrum of H, for the systems of physical interest, is bounded from below. Nevertheless, it will be instructive to consider first the case of an unbounded continuous spectrum.

3.1. *An infinite continuous spectrum*

Suppose that the continuous spectrum of H is infinite, $\mathscr{I} = (-\infty; +\infty)$. The survival probability then may be computed exactly without even introducing any form factor. Indeed, although $v(\omega) = 1$ is not square integrable, one may interpret the integral in the RHS of (2.9) as

$$\sigma(z) = \operatorname*{Lim}_{\mu \to +\infty} \int_{-\mu}^{+\mu} \mathrm{d}\omega \frac{1}{\omega - z} = \pm \mathrm{i}\pi \tag{3.1}$$

where the $+(-)$ sign holds for $\mathrm{Im}(z) > 0$ ($\mathrm{Im}(z) < 0$) and so

$$\eta(z) = \eta_\pm(z) = \omega_0 - z \mp \mathrm{i}\pi\lambda^2 \tag{3.2}$$

with the same convention for the + or − signs as before. Recalling that the contour $\bar{C}$ in (2.6) lies above the real axis for $t \geqslant 0$, the survival probability (for the spontaneous emission problem) decays exponentially ($t \geqslant 0$)

$$\begin{aligned} p(t) = |f_0(t)|^2 &= \left| \frac{1}{2\pi \mathrm{i}} \int_{\bar{C}} \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}zt} \frac{1}{\eta_+(z)} \right|^2 \\ &= |\exp[-\mathrm{i}(\omega_0 - \mathrm{i}\pi\lambda^2)t]|^2 = \exp(-2\pi\lambda^2 t). \end{aligned} \tag{3.3}$$

Consider now the Schrödinger equation (2.5). It takes the form

$$\mathrm{i}\partial_t f_0(t) = \omega_0 f_0(t) + \lambda \int_{-\infty}^{+\infty} \mathrm{d}\omega\, f(\omega;t) \tag{3.4}$$

$$\mathrm{i}\partial_t f(\omega;t) = \lambda f_0(t) + \omega f(\omega;t). \tag{3.5}$$

As $\{f_0(t=0); f(\omega; t=0)\} = \{1; 0\}$ for the spontaneous emission problem, the solution of (3.5) is

$$f(\omega;t) = -\mathrm{i}\lambda\, \mathrm{e}^{-\mathrm{i}\omega t} \int_0^t \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}\omega\tau} f_0(\tau). \tag{3.6}$$

Replacing this expression for $f(\omega;t)$ in (3.4), yields

$$\mathrm{i}\partial_t f_0(t) = \omega_0 f_0(t) - \mathrm{i}\lambda^2 \int_{-\infty}^{+\infty} \mathrm{d}\omega\, \mathrm{e}^{-\mathrm{i}\omega t} \int_0^{\tau} \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}\omega\tau} f_0(\tau) \tag{3.7}$$

and the solution of (3.7), such that $f_0(t=0)=1$, is:

$$f_0(t)=\begin{cases}\exp[-\mathrm{i}(\omega_0-\mathrm{i}\pi\lambda^2)t] & (t\geqslant 0)\\ \exp[-\mathrm{i}(\omega_0+\mathrm{i}\pi\lambda^2)t] & (t\leqslant 0)\end{cases}. \tag{3.8}$$

We now introduce a form factor, and we show that the α-dependent survival probability $p(t;\,\alpha)$ tends to the correct result (3.3), by taking the limit $\alpha\to+\infty$.

Therefore, consider a Lorentzian form factor centred on $\omega=\omega_0$

$$\chi(\omega;\,\alpha)=\alpha^2/[(\omega-\omega_0)^2+\alpha^2]. \tag{3.9}$$

Our model then coincides with the one considered by Pietenpol (1967). A straightforward calculation yields

$$\sigma(z;\,\alpha)=\int_{-\infty}^{+\infty}\mathrm{d}\omega\,\frac{1}{\omega-z}\,\frac{\alpha^2}{(\omega-\omega_0)^2+\alpha^2}=\sigma_\pm(z;\,\alpha) \tag{3.10}$$

with

$$\sigma_\pm(z;\,\alpha)=-\pi\alpha/(z-\omega_0\pm\mathrm{i}\alpha) \tag{3.11}$$

where the $+(-)$ sign holds for $\mathrm{Im}(z)>0$ ($\mathrm{Im}(z)<0$), and

$$\eta(z;\,\alpha)=\eta_\pm(z;\,\alpha)=\omega_0-z+\pi\lambda^2\alpha/(z-\omega_0\pm\mathrm{i}\alpha) \tag{3.12}$$

with the same convention for the + or − signs as already stated. For any value of $\lambda(\neq 0)$, the function $\eta_+(z;\,\alpha)$ has two complex zeros in the lower half complex plane

$$z_0(\lambda;\,\alpha)=\omega_0-\mathrm{i}\alpha/2+\mathrm{i}(\alpha/2)(1-4\pi\lambda^2/\alpha)^{1/2} \tag{3.13}$$

$$z_1(\lambda;\,\alpha)=\omega_0-\mathrm{i}\alpha/2-\mathrm{i}(\alpha/2)(1-4\pi\lambda^2/\alpha)^{1/2}. \tag{3.14}$$

Hence the survival probability for the spontaneous emission problem is given by $p(t;\,\alpha)=|f_0(t;\,\alpha)|^2$ with (see (2.12)), $(t\geqslant 0)$

$$f_0(t;\,\alpha)=[(1+\gamma)\,\mathrm{e}^{-\mathrm{i}z_0t}-(1-\gamma)\,\mathrm{e}^{-\mathrm{i}z_1t}]/2\gamma \tag{3.15}$$

where

$$\gamma=(1-4\pi\lambda^2/\alpha)^{1/2}. \tag{3.16}$$

From (3.15), it is easily seen that $\dot p(t=0;\,\alpha)=0$ (here $\dot p(t;\,\alpha)=\mathrm{d}p(t;\,\alpha)/\mathrm{d}t$). It is possible to remove the α-dependence by taking the limit $\alpha\to+\infty$: (see also Middleton and Schieve 1973)

$$\operatorname*{Lim}_{\alpha\to\infty}p(t;\,\alpha)=\operatorname*{Lim}_{\alpha\to\infty}|f_0(t;\,\alpha)|^2=\exp(-2\pi\lambda^2t). \tag{3.17}$$

So for an infinite spectrum, the removal of the unphysical parameter α leads to the correct result (compare (3.17) and (3.3)).

3.2. *A semi-infinite continuous spectrum*

We now assume the continuous spectrum of H to be semi-infinite, $\mathcal{S}\equiv[-\mu;\,+\infty)$. Here μ is some finite, non-negative real number.

As the complex function $\sigma(z)$ is now ill-defined

$$\sigma(z)=\int_{-\mu}^{+\infty}\mathrm{d}\omega\,\frac{1}{\omega-z}\simeq\ln(+\infty)\qquad(\mu<+\infty), \tag{3.18}$$

a form factor is needed. We introduce as before the Lorentzian form factor defined in (3.9). We have

$$\sigma(z;\alpha)=\int_{-\mu}^{+\infty}\mathrm{d}\omega\,\frac{1}{\omega-z}\,\frac{\alpha^2}{(\omega-\omega_0)^2+\alpha^2}$$
$$=\frac{-\alpha^2}{(z-\omega_0)^2+\alpha^2}[\ln(-\mu-z)-\ln[(\mu+\omega_0)^2+\alpha^2]^{1/2}$$
$$+[(z-\omega_0)/\alpha]\{\pi/2+\tan^{-1}[(\mu+\omega_0)/\alpha]\}] \tag{3.19}$$

where the branches of the ln and of the $\tan^{-1}$ functions are such that $\ln(-\mu-z)=0$ for $z=-\mu-1$ and $|\tan^{-1}(x)|\leq+\pi/2$.

From $\chi(-\mu;\alpha)\neq 0$ it follows that $\sigma(-\mu;\alpha)=+\infty$ and that the function $\eta(z;\alpha)$ (see (2.8) together with (3.19)) admits one real zero, ν_L, for any value of $\lambda(\neq 0)$. Let us isolate the contribution arising from the eigenvalue ν_L in the solution $p(t;\alpha)=|f_0(t;\alpha)|^2$. We have, using Cauchy's residue theorem

$$f_0(t;\alpha)=\frac{\mathrm{e}^{-\mathrm{i}\nu_\mathrm{L}t}}{-\eta'(\nu_\mathrm{L};\alpha)}+\frac{1}{2\pi\mathrm{i}}\int_\Gamma \mathrm{d}z\,\mathrm{e}^{-\mathrm{i}zt}\,\frac{1}{\eta(z;\alpha)}. \tag{3.20}$$

The contour Γ is depicted in figure 1 and $\eta'(\nu;\alpha)=\mathrm{d}\eta(\nu;\alpha)/\mathrm{d}\nu$. If the energy expectation value for the initial state $\{1;0\}$ is finite, then $\dot{p}(t=0;\alpha)=0$ (see Chiu *et al* 1977 and references quoted there). It is also interesting to isolate the exponentially decaying contributions to the survival probability. As we know, these contributions may be obtained by the standard methods of analytic continuation (see for example Marchand 1968, Grecos 1978 and references therein). It is sufficient for our purpose to notice that the cut of $\eta(z;\alpha)$ comes from the term $\ln(-\mu-z)$ in (3.19). The analytic continuation of $\eta(z;\alpha)$ below the cut $[-\mu;+\infty)$ may therefore be obtained by changing the branch of the ln function in such a way that $\ln(-\mu-z)=\ln(\mu+z)-\mathrm{i}\pi$ with $\ln(\mu+z)=0$ for $z=-\mu+1$. This corresponds to a rotation of the cut $[-\mu;+\infty)$ by an angle $\theta=-\pi$ (see figure 2). The analytic continuation of $\eta(z;\alpha)$ below the cut $[-\mu;+\infty)$ is thus

$$\eta_+(z;\alpha)=\omega_0-z-\lambda^2\sigma_+(z;\alpha) \tag{3.21}$$

with

$$\sigma_+(z;\alpha)=\frac{-\alpha^2}{(z-\omega_0)^2+\alpha^2}[\ln(\mu+z)-\ln[(\mu+\omega_0)^2+\alpha^2]^{1/2}$$
$$+[(z-\omega_0)/\alpha]\{\pi/2+\tan^{-1}[(\mu+\omega_0)/\alpha]\}-\mathrm{i}\pi]. \tag{3.22}$$

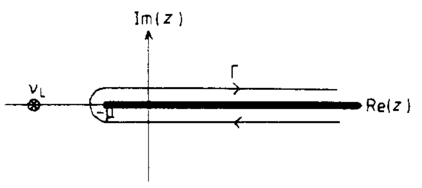

Figure 1. The contour Γ.

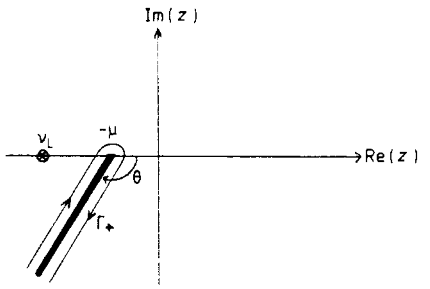

Figure 2. Rotation of the cut by an angle $\theta=\theta_0$.

We note here that the results for an infinite spectrum (see (3.12)) are recovered by taking the limit $\mu \to +\infty$ in the expressions above.

For any $\lambda(\neq 0)$ the analytic continuation $\eta_+(z;\alpha)$ has at least two complex zeros in the lower half complex plane. As before, we denote by $z_0(\lambda;\alpha)$ and by $z_1(\lambda;\alpha)$ the zeros which tend to ω_0 and to $\omega_0 - \mathrm{i}\alpha$ when $\lambda \to +0$, respectively. The solution $f_0(t;\alpha)$ (see (3.20)) may be expressed as

$$f_0(t;\alpha) = \frac{\mathrm{e}^{-\mathrm{i}\nu_L t}}{-\eta'(\nu_L;\alpha)} + \frac{\mathrm{e}^{-\mathrm{i}z_0 t}}{-\eta'_+(z_0;\alpha)} + \frac{\mathrm{e}^{-\mathrm{i}z_1 t}}{-\eta'_+(z_1;\alpha)} + \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_+} \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}zt}\frac{1}{\eta_+(z;\alpha)}. \tag{3.23}$$

The contour Γ_+ encloses only the rotated cut (see figure 2). Here we have assumed implicitly that $\eta_+(z;\alpha)$ has *only* two complex zeros (z_0 and z_1) in the lower half plane. Comparison with (3.15) shows also that the first term (the contribution arising from ν_L) and the last (the integral) one vanish at the limit $\mu \to +\infty$.

Some analytical results concerning the removal of the unphysical parameter α may be obtained if one notices that, for large values of it, the complex function $\sigma(z;\alpha)$ given by (3.19) behaves like $-\ln[(-\mu - z)/\alpha]$ near the lower end-point $z = -\mu$ of $\mathcal{I}$. As we expect that the main contribution to the integral in the RHS of equation (3.20) comes from those points z which lie not too far from the lower end-point $-\mu$ of $\mathcal{I}$, we approximate $\sigma(z;\alpha)$ by $\tilde{\sigma}(z;\alpha)$, such that

$$\tilde{\sigma}(z;\alpha) = -\ln[(-\mu - z)/\alpha]. \tag{3.24}$$

At the end of this section, we shall give some other examples of form factors which lead to the same approximation. $\tilde{\sigma}(z;\alpha)$ is thus independent of the precise choice of the form factor if $\chi(\omega = -\mu;\alpha) \neq 0$. The only reason why we introduce such an approximation is that it permits us, as we shall see below, to derive some simple analytical results. In the next section we shall see, by numerical means, that the discrepancies between the exact and approximate results (respectively computed by using the exact $\sigma(z;\alpha)$ and the approximate $\tilde{\sigma}(z;\alpha)$ function) disappear for sufficiently large values of α.

So the approximate survival probability for the spontaneous emission problem is given by ($t \geq 0$)

$$\tilde{p}(t;\alpha) = |\tilde{f}_0(t;\alpha)|^2 = \left| \frac{1}{2\pi\mathrm{i}} \int_{\tilde{C}} \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}zt} \frac{1}{\tilde{\eta}(z;\alpha)} \right|^2 \tag{3.25}$$

where $\tilde{C}$ is a straight line parallel to the real axis from above, and

$$\tilde{\eta}(z;\alpha) = \omega_0 - z - \lambda^2 \tilde{\sigma}(z;\alpha). \tag{3.26}$$

As $\tilde{\sigma}(z = -\mu;\alpha) = +\infty$, the function $\tilde{\eta}(z;\alpha)$ has one real zero, $\tilde{\nu}_L$, for any finite value of $\lambda(\neq 0)$. One may isolate again the contribution arising from $\tilde{\nu}_L$. We have

$$\tilde{f}_0(t;\alpha) = \frac{\mathrm{e}^{-\mathrm{i}\tilde{\nu}_L t}}{-\tilde{\eta}'(\tilde{\nu}_L;\alpha)} + \frac{1}{2\pi\mathrm{i}} \int_{\Gamma} \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}zt} \frac{1}{\tilde{\eta}(z;\alpha)}, \tag{3.27}$$

the contour Γ being drawn in figure 1.

The analytic continuation of $\tilde{\eta}(z;\alpha)$ below the cut $[-\mu; +\infty)$ is performed in the same way as that of $\eta(z;\alpha)$. We have (compare also with (3.21) and (3.22))

$$\tilde{\eta}_+(z;\alpha) = \omega_0 - z - \lambda^2 \tilde{\sigma}_+(z;\alpha) \tag{3.28}$$

with

$$\tilde{\sigma}_+(z;\,\alpha) = -\{\ln[(\mu+z)/\alpha] - \mathrm{i}\pi\} \tag{3.29}$$

where $\ln(\mu+z)=0$ for $z=-\mu+1$. The complex function $\tilde{\eta}_+(z;\,\alpha)$ has at least one complex zero, $\tilde{z}_0(\lambda;\,\alpha)$, in the lower half complex plane, and $\tilde{z}_0(\lambda;\,\alpha)\to+\omega_0$ when $\lambda\to+0$. Therefore we have (compare with (3.23))

$$\tilde{f}_0(t;\,\alpha) = \frac{\mathrm{e}^{-\mathrm{i}\tilde{\nu}_{\mathrm{L}}t}}{-\tilde{\eta}'(\tilde{\nu}_{\mathrm{L}};\,\alpha)} + \frac{\mathrm{e}^{-\mathrm{i}\tilde{z}_0 t}}{-\tilde{\eta}'_+(\tilde{z}_0;\,\alpha)} + \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_+} \mathrm{d}z\,\mathrm{e}^{-\mathrm{i}zt}\frac{1}{\tilde{\eta}_+(z;\,\alpha)} \tag{3.30}$$

where the contour Γ_+ encloses only the rotated cut (see figure 2). We assume here that $\tilde{\eta}_+(z;\,\alpha)$ has only one complex zero ($\tilde{z}_0$) in the lower half plane.

In order to remove the cut-off parameter α, we proceed in two stages. First, we show that the amplitude of probability $[-\tilde{\eta}'(\tilde{\nu}_{\mathrm{L}};\,\alpha)]^{-1}$ tends, at the limit $\alpha\to+\infty$, to unity. Second, we show that the integral on the RHS of (3.27) vanishes when $\alpha\to+\infty$. We prove our first statement as follows. Let the cut-off parameter be given by ($\lambda\neq0$)

$$\alpha=\alpha(n)=n\lambda^2\exp[(\omega_0+\mu+n\lambda^2)/\lambda^2]\to+\infty \text{ when } n\to+\infty. \tag{3.31}$$

Here n represents some real positive number. Using the equation $\tilde{\eta}(\tilde{\nu}_{\mathrm{L}};\,\alpha(n))=0$, we get

$$\tilde{\nu}_{\mathrm{L}}=\nu(n)=-\mu-n\lambda^2\to-\infty \text{ when } n\to+\infty. \tag{3.32}$$

Hence it follows that

$$[-\tilde{\eta}'(\nu(n);\,\alpha(n))]^{-1}=(1+1/n)^{-1}\to1 \text{ when } n\to+\infty \tag{3.33}$$

which is the announced result. We have mentioned as well (below (3.23)) that the contribution arising from the eigenvalue vanishes at the limit $\mu\to+\infty$ (and for $\alpha<+\infty$). From geometrical considerations (see figure 3), we get

$$-\mu>\tilde{\nu}_{\mathrm{L}}>\tilde{\nu}_{\mathrm{M}}=-\mu-\alpha\exp[(-\omega_0-\mu)/\lambda^2], \tag{3.34}$$

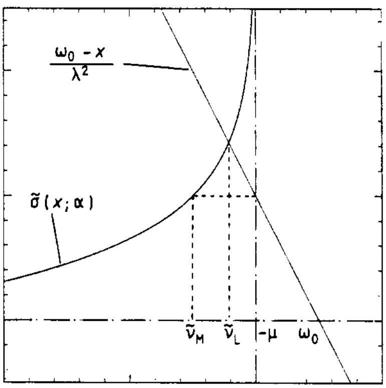

Figure 3. Geometrical determination of the parameter $\tilde{\nu}_{\mathrm{M}}$.

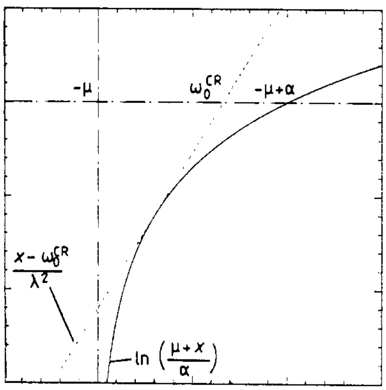

Figure 4. Geometrical determination of the parameter ω_0^{CR}.

thus

$$0 \leqslant [-\tilde{\eta}'(\tilde{\nu}_L; \alpha)]^{-1} < \{1 + (\lambda^2/\alpha) \exp[(\omega_0 + \mu)/\lambda^2]\}^{-1} \tag{3.35}$$

and our assertion follows trivially.

Consider now the integral in the RHS of (3.27). Using the relation $\ln(-\mu - x \pm \mathrm{i}0) = \ln(\mu + x) + \mathrm{i}\pi$ for every $x \in [-\mu; +\infty)$, we rewrite it in the form

$$I(t; \alpha) = \int_{-\mu}^{+\infty} \mathrm{d}x \, \mathrm{e}^{-\mathrm{i}xt} h(x; \alpha) \tag{3.36}$$

with

$$h(x; \alpha) = \lambda^2 / [\![\{\omega_0 - x + \lambda^2 \ln[(\mu + x)/\alpha]\}^2 + (\pi\lambda^2)^2]\!]. \tag{3.37}$$

The function $h(x; \alpha)$ vanishes for $x = -\mu$ and $x = +\infty$. Its extrema are the solutions of

$$0 = -1 + \lambda^2/(\mu + x) \tag{3.38}$$

or of

$$0 = \omega_0 - x + \lambda^2 \ln[(\mu + x)/\alpha]. \tag{3.39}$$

The last equation admits no solutions if $\omega_0 < \omega_0^{\mathrm{CR}} \Leftrightarrow \alpha > \alpha_{\mathrm{CR}}$, where

$$\omega_0^{\mathrm{CR}} = -\mu + \lambda^2 - \lambda^2 \ln(\lambda^2/\alpha) \tag{3.40}$$

and

$$\alpha_{\mathrm{CR}} = \lambda^2 \exp[(\mu + \omega_0 - \lambda^2)/\lambda^2]. \tag{3.41}$$

The parameter ω_0^{CR} is fixed by noticing that $1/\lambda^2$ is the slope of $\ln[(\mu + x)/\alpha]$ for $x = -\mu + \lambda^2$. A glance at figure 4 clearly shows that (3.39) admits no solutions if $\omega_0 < \omega_0^{\mathrm{CR}}$ or equivalently if $\alpha > \alpha_{\mathrm{CR}}$; α_{CR} is fixed by the relation $\omega_0 = \omega_0^{\mathrm{CR}}(\alpha = \alpha_{\mathrm{CR}})$, that is by (3.41).

For $\alpha > \alpha_{\mathrm{CR}}$, and for any finite value of $\lambda (\neq 0)$, the function $h(x; \alpha)$ has a single maximum at $x = -\mu + \lambda^2$ (see (3.38)). Then we have $0 \leqslant h(x; \alpha) \leqslant h(-\mu + \lambda^2; \alpha)$ for every $x \in [-\mu; +\infty)$. Hence it follows that

$$|I(t; \alpha)| \leqslant \frac{\lambda^2(\omega_0^{\mathrm{CR}} + \mu)}{(\omega_0 - \omega_0^{\mathrm{CR}})^2 + (\pi\lambda^2)^2} + \int_{\omega_0^{\mathrm{CR}}}^{+\infty} \mathrm{d}x \, h(x; \alpha). \tag{3.42}$$

Finally as $\omega_0^{\mathrm{CR}} \to +\infty$ for $\alpha \to +\infty$, we get $(0 \neq \lambda^2 < +\infty;\ \mu < +\infty)$

$$\operatorname*{Lim}_{\alpha \to +\infty} |I(t; \alpha)| = 0. \tag{3.43}$$

Since (by assumption) $\tilde{\eta}_+(z; \alpha)$ has only one complex zero $\tilde{z}_0(\lambda; \alpha)$ in the lower half plane, this result means in particular that the contribution arising from the rotated cut (see (3.30)) compensates exactly (at the limit $\alpha \to +\infty$) for the one arising from the complex zero $\tilde{z}_0(\lambda; \alpha)$.

We finally obtain, using (3.33) and (3.43) $(0 \neq \lambda^2 < +\infty;\ \mu < +\infty)$

$$\operatorname*{Lim}_{\alpha \to +\infty} \tilde{p}(t; \alpha) = \operatorname*{Lim}_{\alpha \to +\infty} |\exp(-\mathrm{i}\tilde{\nu}_L t)|^2 = 1. \tag{3.44}$$

Thus, if the continuous spectrum is bounded from below, the approximate survival probability (for the spontaneous emission problem) does not undergo any decay at the limit $\alpha \to +\infty$. This rather surprising result contrasts with the case of an infinite continuous spectrum (recall (3.3) or (3.17)) and it shows that the two limits $\mu \to +\infty$

and $\alpha \to +\infty$ may not be permuted. We note that the case of a 'constant' form factor

$$\chi(\omega;\alpha)=\begin{cases}1 & \text{for } \omega\in[-\mu;-\mu+\alpha]\\ 0 & \text{for } \omega\notin[-\mu;-\mu+\alpha]\end{cases} \tag{3.45}$$

can be treated along similar lines, except that no approximation is even necessary. Although the Hamiltonian has then two eigenvalues (ν_L and ν_R) for any finite value of $\lambda(\neq 0)$, we obtain similar analytical results: at the limit $\alpha\to+\infty$, $[-\eta'(\nu_L;\alpha)]^{-1}$ tends to 1, and the contributions arising from ν_R and from the cut $[-\mu;-\mu+\alpha]$ vanish. We then get the exact result $(0\neq\lambda^2<+\infty;\ \mu<+\infty)$

$$\lim_{\alpha\to+\infty} p(t;\alpha)=\lim_{\alpha\to+\infty}|\exp(-\mathrm{i}\nu_L t)|^2=1. \tag{3.46}$$

We have omitted these latter calculations here because of their likeness to the previous ones. However, note that the same kind of arguments as those used to derive (3.35) may be used to prove that the contribution arising from ν_R vanishes at the limit $\alpha\to+\infty$.

We now illustrate, by means of some new examples, the fact that the approximate complex function $\tilde\sigma(z;\alpha)$, given by (3.24), is independent of the precise choice of the form factor $\chi(\omega;\alpha)$, if $\chi(\omega=-\mu;\alpha)\neq 0$. For the constant form factor above, a straightforward calculation yields

$$\begin{aligned}\sigma(z;\alpha)&=\int_{-\mu}^{-\mu+\alpha}\mathrm{d}\omega\frac{1}{\omega-z}=\ln[1-\alpha/(\mu+z)]\\ &=\tilde\sigma(z;\alpha)+\ln[1-(\mu+z)/\alpha]\end{aligned} \tag{3.47}$$

where the second term on the RHS is negligible for small values of $(\mu+z)/\alpha$. For an exponential form factor, $\chi(\omega;\alpha)=\exp[-(\omega+\mu)/\alpha]$, we get

$$\begin{aligned}\sigma(z;\alpha)&=\int_{-\mu}^{+\infty}\mathrm{d}\omega\frac{\exp-(\omega+\mu)/\alpha}{\omega-z}\\ &=\exp[-(z+\mu)/\alpha]E_1[(-\mu-z)/\alpha]\end{aligned} \tag{3.48}$$

where $E_1(z)$ is the exponential integral (Abramowitz and Stegun 1964). Using the expansion for small z

$$E_1(z)=-\gamma-\ln(z)-\sum_{n=1}^{\infty}\frac{(-1)^n z^n}{n n!}\qquad (|\arg z|<\pi) \tag{3.49}$$

where γ is Euler's constant, it is readily seen that $\tilde\sigma(z;\alpha)$ is again the leading term of an expansion of $\sigma(z;\alpha)$ around $z=-\mu$, and for large values of α. Note that for the (three) form factors under consideration here, $\chi(-\mu;\alpha)\neq 0$ and $\sigma(-\mu;\alpha)=+\infty=\tilde\sigma(-\mu;\alpha)$. If we consider a form factor such that $\chi(-\mu;\alpha)=0$, then $\sigma(-\mu;\alpha)<+\infty$ and $\tilde\sigma(z;\alpha)$ is no longer a 'good' approximation, and for sufficiently small values of λ, H has no eigenvalues. Nevertheless, it might be seen analytically and/or numerically that (3.46) still remains true.

We end this section with the following remark. One of the central quantities which comes into play in non-equilibrium statistical mechanics is the collision operator $\psi(z\to+\mathrm{i}0)$ (see for example Prigogine and Grecos 1979). For the one-point Friedrichs model, $\psi(z)$ is simply a function and is given, up to second order in λ, by (Grecos and Prigogine 1972)

$$\psi(z)=-\lambda^2[\sigma(\omega_0+z)-\sigma(\omega_0-z)]. \tag{3.50}$$

We omitted here the dependence on α. If the complex function $\sigma(z)$ is given by its approximate form, (3.24), with $\mu = 0$, we get

$$\psi(z) = -\lambda^2 \ln[(\omega_0 - z)/(\omega_0 + z)] \tag{3.51}$$

which is precisely the expression derived by Davidson and Kozak (1970) (up to an inessential factor ω_0 in front of the RHS of (3.51)).

4. Numerical results

This section is devoted to a detailed numerical investigation of the survival probability for the spontaneous emission problem. We are mainly interested in its dependence on: (a) the cut-off parameter α; (b) the lower end-point $-\mu$ of the continuous spectrum; and (c) the coupling parameter λ.

We observed in the previous section that, for the case of a constant coupling function $v(\omega) = 1$, it was interesting to introduce a Lorentzian form factor, centred on $\omega = \omega_0$, to remove some infinity (recall (3.18)), the reason is that the survival probability may be computed exactly if the continuous spectrum is infinite (see (3.15)). So our numerical results for a semi-infinite continuous spectrum may be compared with the exact analytical results for the infinite spectrum. In this way, we shall analyse in great detail the influence of the lower end-point of the continuous spectrum. We note that the case of a Lorentzian form factor was not considered by Yang *et al* (1974).

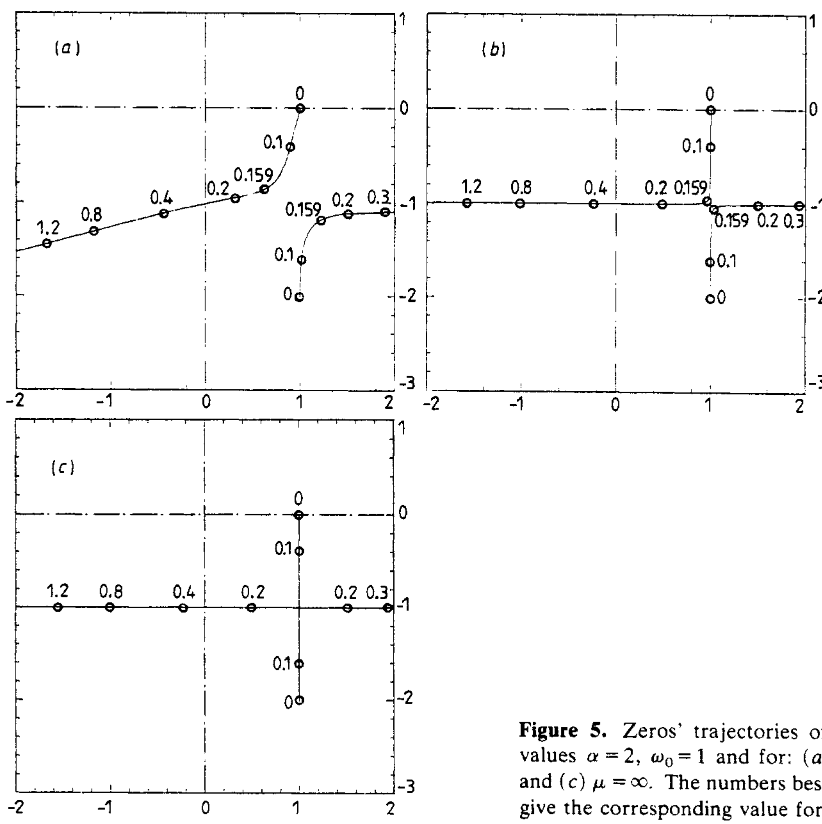

Figure 5. Zeros' trajectories of $\eta_+(z;\alpha)$ for the values $\alpha = 2$, $\omega_0 = 1$ and for: (a) $\mu = 0$; (b) $\mu = 10$ and (c) $\mu = \infty$. The numbers beside the open circles give the corresponding value for λ^2.

Analytical results concerning the removal of the unphysical parameter α have also been obtained in the previous section. As the explicit expression for $\sigma(z;\alpha)$, the form factor being that Lorentzian centred on $\omega=\omega_0$, is too complicated, an approximation, $\tilde{\sigma}(z;\alpha)$, has been introduced. We mentioned, however, that the discrepancies between the exact and approximate results disappear for sufficiently large values of α. This will be illustrated numerically in this section.

We mentioned (below (3.23)) that the contributions to the survival probability, arising from the eigenvalue ν_L and from the rotated cut (the remainder integral on the RHS of (3.23)), vanish at the limit $\mu\to+\infty$. So, for sufficiently large values of μ, the main contribution to the survival probability comes from z_0 and z_1. For $\mu=+\infty$, these complex zeros are given by (3.13) and (3.14). Our first series of numerical results illustrates how the zeros' trajectories of $\eta_+(z;\alpha)$ (given by (3.21) with (3.22)) depend on the lower end-point $-\mu$ of the continuous spectrum $\mathscr{S}$. For a given value of α and for $\mu=+\infty$, z_0 and z_1 move along a straight line parallel to the imaginary axis for $\lambda^2<\alpha/4\pi$, and parallel to the real axis for $\lambda^2>\alpha/4\pi$; they coalesce at the point $z=\omega_0-\mathrm{i}\alpha/2$ for $\lambda^2=\alpha/4\pi$. Once μ is finite, they do not coalesce for any value of λ (see figure 5). If however μ is sufficiently large, they come close to the point $z=\omega_0-\mathrm{i}\alpha/2$ and still move along a straight line parallel to the imaginary axis for small values of λ ($\lambda^2<\alpha/4\pi$) and parallel to the real axis for large values of λ ($\lambda^2>\alpha/4\pi$). A glance at figure 5 shows also that it is the trajectory of $z_0(\lambda;\alpha)$ (recall that $z_0(\lambda;\alpha)$ is the complex zero which tends to ω_0 when $\lambda\to+0$) which undergoes the main deformation. If now λ is kept fixed and α varies, then, for $\mu=+\infty$, z_0 and z_1 move along a circle until they coalesce at the point $z=\omega_0-\mathrm{i}\alpha/2$ for $\alpha=4\pi\lambda^2$. Once $\alpha>4\pi\lambda^2$, they move along a straight line parallel to the imaginary axis and, at the limit $\alpha\to+\infty$, $z_0(\lambda;\alpha)$ and $z_1(\lambda;\alpha)$ tend to $\omega_0-\mathrm{i}\pi\lambda^2$ and to $\omega_0-\mathrm{i}\infty$, respectively. For $\mu<+\infty$, it is again the trajectory of $z_0(\lambda;\alpha)$ which undergoes the main deformation (see figure 6): the most striking feature is that $z_0(\lambda;\alpha)$ tends to $-\infty-2\mathrm{i}\pi\lambda^2$ at the limit $\alpha\to+\infty$. For $\mu=0$, this is clearly shown in figure 6. For $\mu=10$, $\alpha\simeq10^8$ is not yet sufficiently large to see that $\mathrm{Lim}_{\alpha\to\infty}\,\mathrm{Im}(z_0)=-2\pi\lambda^2$. Indeed for large values of α, we have $z_0\simeq\tilde{z}_0$ and $\tilde{z}_0$ is the complex zero of $\tilde{\eta}_+(z;\alpha)$. Solving $\tilde{\eta}_+(z;\alpha)=0$ iteratively yields

$$
\begin{aligned}
\tilde{z}_0^{(0)} &= \omega_0\\
\tilde{z}_0^{(1)} &= \omega_0+\lambda^2\ln[(\mu+\omega_0)/\alpha]-\mathrm{i}\pi\lambda^2\\
\tilde{z}_0^{(2)} &= \omega_0+\lambda^2\ln[(\mu+\tilde{z}_0^{(1)})/\alpha]-\mathrm{i}\pi\lambda^2.\\
&\vdots
\end{aligned}
$$

If $\mu+\mathrm{Re}(\tilde{z}_0^{(1)})<0$, this is if $\alpha>\tilde{\alpha}_{\mathrm{CR}}\equiv(\mu+\omega_0)\exp[(\mu+\omega_0)/\lambda^2]$, then $\tilde{z}_0^{(2)}\to-\infty-2\mathrm{i}\pi\lambda^2$ when $\alpha\to+\infty$. On the other hand if $\alpha<\tilde{\alpha}_{\mathrm{CR}}$ then $\tilde{z}_0^{(2)}\to+\infty-\mathrm{i}\pi\lambda^2$ when $\alpha\to+0$. For the values $\omega_0=1$ and $\pi\lambda^2=1$ used in figure 6, we have $\tilde{\alpha}_{\mathrm{CR}}=\exp(\pi)\simeq23$ if $\mu=0$, and $\tilde{\alpha}_{\mathrm{CR}}=11\exp(11\pi)\simeq10^{16}$ if $\mu=10$. So for $\mu=10$, we have to consider values of α larger than 10^{16} to see that $\mathrm{Lim}_{\alpha\to\infty}\,\mathrm{Im}(z_0)=-2\pi\lambda^2$.

One of the main analytical results we obtained in the previous section is illustrated in figure 7: $\mathrm{Lim}_{\alpha\to\infty}[-\eta'(\nu_L;\alpha)]^{-1}=1$, independent of $\lambda(\neq0)$. In figure 8, we have plotted the time dependence of the exact and the approximate survival probabilities, for the spontaneous emission problem. The numerical computation of the exact solution $p(t;\alpha)$ is based upon (3.20) with (2.10) and (3.19); for the approximate solution $\tilde{p}(t;\alpha)$, we used (3.27). Our numerical results clearly show that, for $\lambda\neq0$, deviations between $p(t;\alpha)$ and $\tilde{p}(t;\alpha)$ become negligible once α is large enough. Of course this

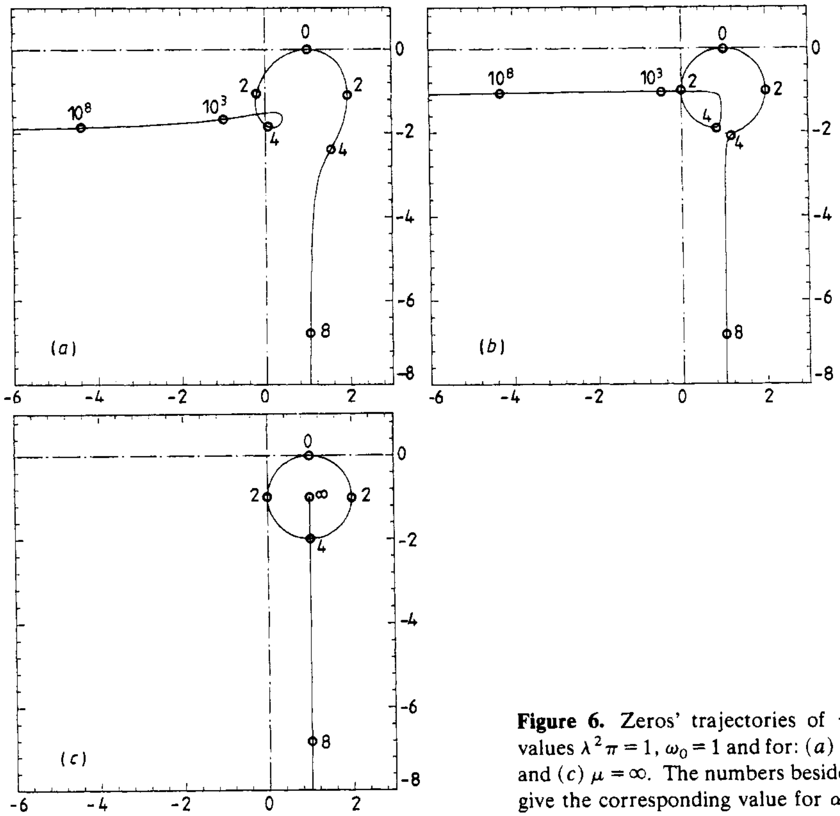

Figure 6. Zeros' trajectories of $\eta_+(z;\alpha)$ for the values $\lambda^2\pi=1$, $\omega_0=1$ and for: (a) $\mu=0$; (b) $\mu=10$ and (c) $\mu=\infty$. The numbers beside the open circles give the corresponding value for α.

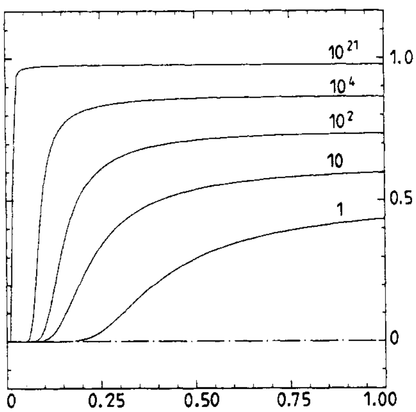

Figure 7. The exact amplitude coefficient $(-\eta'(\nu_L;\alpha))^{-1}$ is plotted against λ^2 and for the values $\omega_0=1$, $\mu=0$ and for five values of α $(1, 10, 10^2, 10^4$ and $10^{21})$.

is due to the fact that both $p(t;\alpha)$ and $\tilde{p}(t;\alpha)$ tend to unity (independent of t and λ) as $\alpha\to+\infty$. This latter result is illustrated, for $p(t;\alpha)$ only, in figure 9. We note here that the relation $\alpha=\alpha(n)$, given by (3.31), is very useful for our numerical computations because it tells us how large α must be in order to get (numerically) $p(t;\alpha)\simeq 1$. It is indeed well known that the integral in the RHS of (3.20) vanishes as $t\to+\infty$ at least

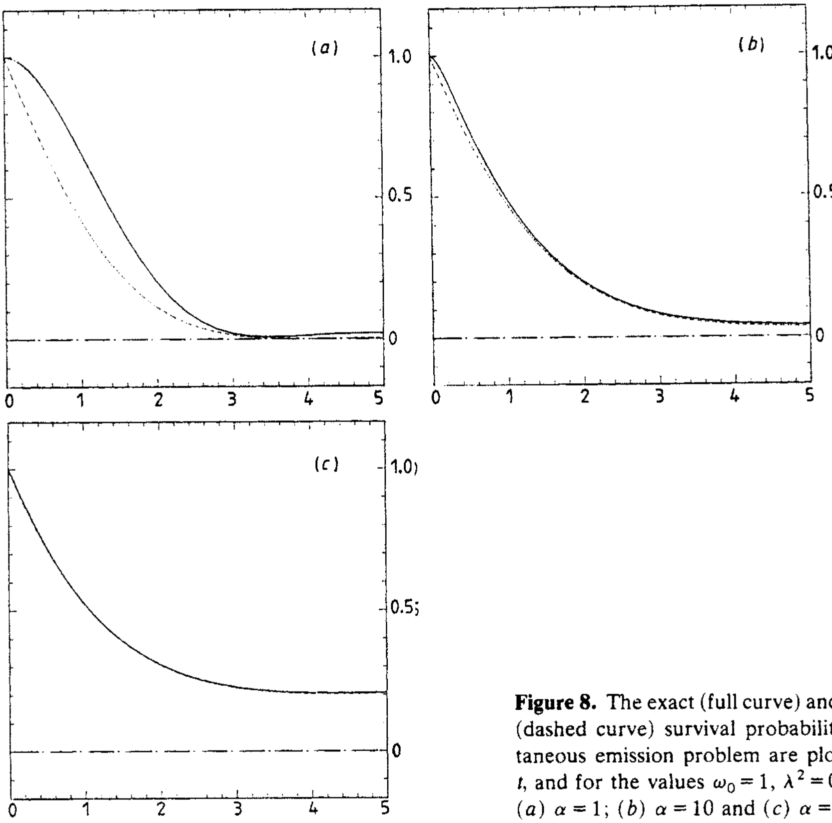

Figure 8. The exact (full curve) and the approximate (dashed curve) survival probabilities for the spontaneous emission problem are plotted against time t, and for the values $\omega_0 = 1$, $\lambda^2 = 0.2$, $\mu = 0$ and for (a) $\alpha = 1$; (b) $\alpha = 10$ and (c) $\alpha = 10^2$.

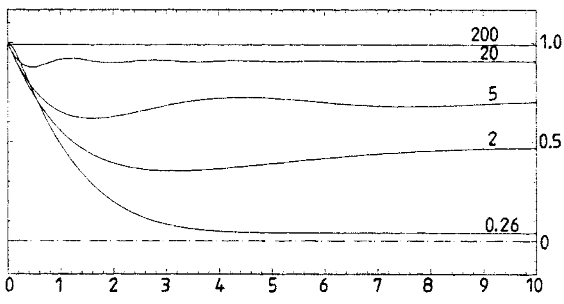

Figure 9. The survival probability $p(t;\, \alpha)$ for the spontaneous emission problem is plotted against time t and for the values $\omega_0 = 1$, $\lambda^2 = 0.2$, $\mu = 0$ and for five values of α given by (3.31) with $n = 0.26$, 2, 5, 20 and 200.

as fast as t^{-1}: this estimate may be obtained integrating by parts. So, as $t \to +\infty$, we have $p(t;\, \alpha) \simeq (-\eta'(\nu_L;\, \alpha))^{-2}$. On the other hand, the exact and approximate solutions are nearly equal for large values of α, and $(-\tilde{\eta}'(\tilde{\nu}_L;\, \alpha)) = 1 + 1/n$ if α is given by (3.31). Notice that $\alpha(n)$ increases exponentially with μ.

The dependence of the exact solution $p(t;\, \alpha)$ on the coupling parameter λ is depicted in figure 10. For $\omega_0 = 1$, $\alpha = 10$ and $\mu = 0$, typical values for the coupling parameter are (see figure 7): $\lambda^2 = 0.1, 0.2$ and 1. For $\lambda^2 = 0.1$, one observes (figure 10) that $p(t;\, \alpha) \to +0$ as $t \to +\infty$; the contribution arising from ν_L does not yet influence

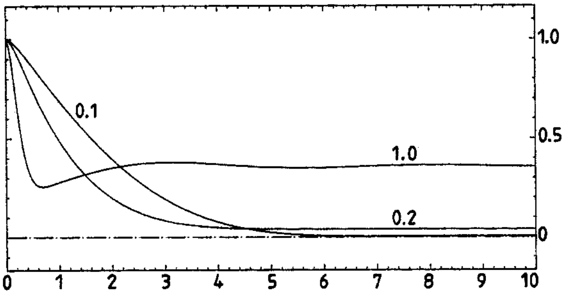

Figure 10. The survival probability $p(t;\,\alpha)$ for the spontaneous emission problem is plotted against time t and for the values $\omega_0 = 1$, $\alpha = 10$, $\mu = 0$ and for three typical values of the coupling parameter: $\lambda^2 = 0.1$, 0.2 and 1.0.

appreciably the asymptotic behaviour of the solution. Recall that ν_L exists for any value of $\lambda(\neq 0)$. For these values of the parameters, one may characterise weak coupling by values of λ such that $\lambda^2 \leqslant 0.1$. For an infinite continuous spectrum, weak coupling may be characterised by values for λ such that $4\pi\lambda^2/\alpha \leqslant 1$. The solution $p(t;\,\alpha) = |f_0(t;\,\alpha)|^2$ then decreases monotonically to zero (see (3.15)). On the contrary, for strong coupling $(4\pi\lambda^2/\alpha > 1)$, the solution exhibits a damped oscillatory decay. For typical weak and strong coupling, we finally illustrate, in figure 11, the dependence of the solution $p(t;\,\alpha)$ on the lower end-point $-\mu$ of the continuous spectrum $\mathcal{S}$. It is clear from figure 11 that, in the 'weak coupling regime', the solution for a semi-infinite spectrum and the one for an infinite spectrum are nearly equal; the 'background' term is then very small, i.e. the contributions arising from the eigenvalue ν_L and from the rotated cut Γ_+ (see (3.23)).

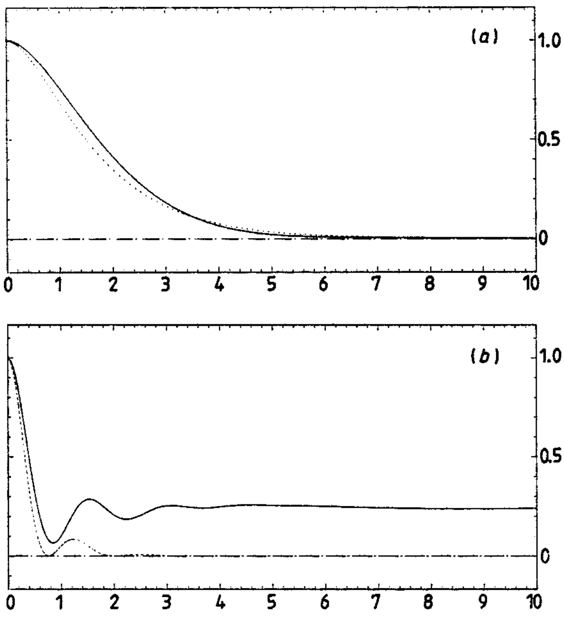

Figure 11. The survival probability $p(t;\,\alpha)$ for the spontaneous emission problem is plotted against time t and for the values $\omega_0 = 1$, $\alpha = 2$, $\mu = 0$ (full curve) or $\mu = +\infty$ (broken curve) and for: (*a*) $\lambda^2 = 0.1$; (*b*) $\lambda^2 = 1.2$.

5. Conclusion

In this paper, we investigated the one-point Friedrichs model (the unperturbed eigenvalue ω_0 of H_0 being embedded into the continuous part of the spectrum) characterised by a λV interaction with a non-square integrable coupling function $v(\omega) = 1$.

If the continuous spectrum of H is infinite, an appropriate interpretation of some integral (see (3.1)) leads to (some) finite results; no form factor is needed (to solve the Schrödinger equation), and the survival probability (for the spontaneous emission problem) decays exponentially: $p(t) = \exp(-2\pi\lambda^2 t)$. If a Lorentzian form factor centred on $\omega = \omega_0$ is now introduced, the removal of the unphysical cut-off parameter α, in the now α-dependent solution $p(t; \alpha)$, leads to the correct result $p(t)$ given before.

For the systems of physical interest, the continuous spectrum of H is bounded from below, $\mathcal{S} \equiv [-\mu; +\infty)$. A form factor is then needed to avoid some infinity (see (3.18)). For that Lorentzian form factor centred on $\omega = \omega_0$, our previous results are recovered by taking the limit $\mu \to +\infty$. To obtain some simple analytical results about the removal of the cut-off parameter α, the exact complex function $\sigma(z; \alpha)$ (which is needed to compute $p(t; \alpha)$) is approximated by $\tilde{\sigma}(z; \alpha) = -\ln[(-\mu - z)/\alpha]$. By means of some examples, we showed that this approximation is independent of the precise choice of the form factor $\chi(\omega; \alpha)$, if $\chi(\omega = -\mu; \alpha) \neq 0$. Due to the existence of an eigenvalue ν_L of H (for any $\lambda \neq 0$, and for $\mu < +\infty$), the limit $\alpha \to +\infty$ of the solution for the spontaneous emission problem remains equal to its initial value $p(t = 0; \alpha) = 1$.

One of the central quantities in non equilibrium mechanics is the collision operator $\psi(z \to +\mathrm{i}0)$. For the model under consideration, $\psi(z)$ reduces to a simple function. To second order in the coupling parameter λ, the mentioned approximation leads to the same function $\psi(z)$ as the one derived by Davidson and Kozak (1970).

We presented also an extensive numerical analysis of the solution of the spontaneous emission problem. Its dependence on the parameters of the problem (i.e. α, μ and λ) has been studied in great detail. The most striking feature is that the complex zero z_0 ($\to \omega_0$ when $\lambda \to +0$) of $\eta_+(z; \alpha)$ (see (3.21) and (3.22)) tends, at the limit $\alpha \to +\infty$, to $-\infty - 2\mathrm{i}\pi\lambda^2$ if $\mu < \infty$, and to $\omega_0 - \mathrm{i}\pi\lambda^2$ if $\mu = \infty$. By computing numerically the exact solution for very large values of α, we illustrated our main result $\mathrm{Lim}_{\alpha \to +\infty}\, p(t; \alpha) = 1$. In the 'weak coupling regime', the solution for a semi-infinite spectrum is nearly equal to the one for an infinite spectrum. In both cases ($\mu < +\infty$ and $\mu = +\infty$) and for any λ, $\dot{p}(t = 0; \alpha) = 0$. However, for $\mu = +\infty$, no form factor is needed and then $\dot{p}(t = 0_\pm) = \pm 2\pi\lambda^2$.

Finally, we note that for an interaction with the kernel $v(\omega) = \omega^{1/2}$, the above approximation leads to the well known problem of ghost states. This will be discussed elsewhere.

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